Chapter 4 Integrals
Exercises Pages 115-116

1. (a) Use the corresponding rules in calculus to establish the following rules when \( w(t) = u(t) + iv(t) \) is a complex-valued function of a real variable \( t \) and \( w'(t) \) exists:
\[
\frac{d}{dt} w(-t) = -w'(-t), \quad \text{where } w'(-t) \text{ denotes the derivative of } w(t) \text{ with respect to } t, \text{ evaluated at } -t.
\]

(b) Use the corresponding rules in calculus to establish the following rules when \( w(t) = u(t) + iv(t) \) is a complex-valued function of a real variable \( t \) and \( w'(t) \) exists:
\[
\frac{d}{dt} |w(t)|^2 = 2w(t)w'(t).
\]

Solution a:
A function \( \omega(t) \) is said to be differentiable at \( t \) when
\[
\frac{d}{dt} \omega(t) = \lim_{\Delta t \to 0} \frac{\omega(t+\Delta t) - \omega(t)}{\Delta t}.
\]
Replacing \( t \) by \(-t\) we get
\[
\frac{d}{dt} \omega(-t) = \lim_{\Delta t \to 0} \frac{\omega(-t+\Delta t) - \omega(-t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\omega(-t+\Delta t) + \omega(-t)}{\Delta t} = -\omega'(t).
\]

Solution b:
\[\omega(t) = u(t) + iv(t) = u + iv\]
\[|\omega(t)|^2 = (u + iv)^2 = u^2 + 2iv + (iv)^2\]
\[
\frac{d}{dt} |\omega(t)|^2 = \frac{d}{dt} (u^2 + 2iv + i^2v^2)
= 2uu' + 2iv' + i^2v'^2
= 2(u' + iv')(u + iv) = 2\omega(t) \cdot \omega'(t).
\]

Solution:
(a) We have \( w'(t) = u'(t) + iv'(t) \) and so \( w'(-t) = u'(-t) + iv'(-t) \). Now we have
\[
\frac{d}{dt} w(-t) = \frac{d}{dt} [u(-t) + iv(-t)]
= \frac{d}{dt} [u(-t)] + i \frac{d}{dt} [v(-t)]
= u'(-t) \frac{d}{dt} (-t) + iv'(-t) \frac{d}{dt} (-t)
= -u'(-t) - iv'(-t) = -[w'(-t) + iv'(-t)] = -w'(-t).
\]

(b) We have \( |w(t)|^2 = [u(t) + iv(t)]^2\)
\[= |u(t)|^2 - |v(t)|^2^2
= [u(t) + iv(t)]^2
= [u(t)]^2 - [v(t)]^2 + i2u(t)v(t).
\]
Therefore
\[
\frac{d}{dt} |w(t)|^2 = \frac{d}{dt} \left\{ [u(t)]^2 - [v(t)]^2 + i2u(t)v(t) \right\}
\]

1
When $n = m$ then the integrand is $1$ and so integral is $\pi/6$. Thus we have seen that $\frac{d}{dp} [w(t)]^2 = 2w(t)w'(t)$.

2. Evaluate the following integrals:

(a) $\int_{1}^{\pi/6} \frac{2}{1-i} \, dt$;

(b) $\int_{0}^{\infty} e^{2it} \, dt$;

(c) $\int_{0}^{\infty} e^{-zt} \, dt \,(\text{Re} \, z > 0)$.

Solution (2b): The derivative of $e^{2it}/2i = e^{i2t}$. Therefore $\int_{0}^{\pi/6} e^{2it} \, dt = \frac{1}{2i} (1/2 + i\sqrt{3}/2 - 1) = i/4 + \sqrt{3}/4$.

Solution (2a): $\int_{1}^{\frac{2}{1-i}} \, dt = \int_{1}^{\frac{1}{1-i}} \, dt = \int_{1}^{\frac{1}{1-i}} \, dt + i\int_{1}^{\frac{2}{1-i}} \, dt = \left(\frac{1}{2} - t\right) \bigg|_{1}^{\frac{1}{2}} = i2\ln(t) \bigg|_{1}^{\frac{1}{2}} = (0) - (-1) = 1$.

Solution (2b): $\int_{0}^{\pi/6} e^{2it} \, dt = \frac{1}{2i} e^{i2\pi/6} = \frac{1}{2i} (e^{i\pi/3} - 1) = \frac{1}{2i} \left(-\frac{1}{2} + i\sqrt{3}/2\right) = \frac{\sqrt{3}}{4} + \frac{i}{4}$.

Solution (2c): $\int_{0}^{\infty} e^{-zt} \, dt = \lim_{N \to \infty} \int_{0}^{N} e^{-zt} \, dt = \lim_{N \to \infty} \frac{1}{-z} e^{-zt} \bigg|_{0}^{N} = \lim_{N \to \infty} \frac{1}{-z} (e^{-zN} - 1)$.

3. Show that if $m$ and $n$ are integers,

$$2\pi \int_{0}^{\theta} e^{im\theta} e^{-in\theta} \, d\theta = \begin{cases} 0 & \text{when } \theta \neq 0, \\ 2\pi & \text{when } \theta = 0 \end{cases}$$

Solution: When $n \neq m$ then an antiderivative is $e^{(m-n)i\theta}/(m-n)i$ and when we plug in $2\pi i$ and $0$ we get the same thing namely, $\frac{1}{m-n}$ and so when we subtract we get 0. When $n = m$ then the integrand is 1 and so integral is $2\pi$.

Solution (3): $2\pi \int_{0}^{\theta} e^{im\theta} e^{-in\theta} \, d\theta = 2\pi \int_{0}^{\theta} e^{i(m-n)\theta} \, d\theta$. If $m = n$, then the right hand side is $2\pi \int_{0}^{\theta} e^{i\theta} = 2\pi$. If $m \neq n$, then the right hand side is equal to $\frac{1}{m-n} e^{i(m-n)\theta/2\pi} - e^{0} = \frac{1}{m-n} (1 - 1) = 0$. 

2
This is a very important integral that occurs frequently in analysis. It gives, e.g., a way to define the Dirac delta function if you are familiar with that from previous studies. The computation is not particularly difficult. We simply use that \( \frac{d}{d\theta} e^{im\theta} = ine^{im\theta} \), \( m \in \mathbb{Z} \). Thus, if \( m \neq n \),
\[
\int_{0}^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_{0}^{2\pi} e^{i(m-n)\theta} d\theta = \frac{2\pi}{i(m-n)} \left[ e^{i(m-n)^2\pi} - 1 \right].
\]
However, since \( m-n \in \mathbb{Z} \), we have \( e^{i(m-n)^2\pi} = 1 \) and thus this integral vanishes. If, however, \( m = n \), then the integral we did above is incorrect as we are dividing by zero. Indeed, in this case, things are even simpler \( \int_{0}^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_{0}^{1} d\theta = 2\pi \).

4. According to definition (2), Section 37, of integrals of complex-valued functions of a real variable, \( \int_{0}^{\pi} e^{i(1+i)x} dx = \int_{0}^{\pi} e^{x} \cos x dx + i \int_{0}^{\pi} e^{x} \sin x dx \). Evaluate the two integrals of the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Solution: This is a nice trick as you may remember that the integrals on the right are rather involved (you have to integrate by parts twice and recognize that you return to where you began but with the opposite side ... thus you can solve the desired equation).

The formula follows easily from the definition of complex exponents and Euler’s formula. Evaluating the left side, we have \( \int_{0}^{\pi} e^{i(1+i)x} dx = \frac{1}{i+1} e^{(1+i)x} \bigg|_{0}^{\pi} = \frac{1}{i+1} \left[ e^{i(1+i)\pi} - 1 \right] = \frac{1}{(1+i)} (e^{\pi} \cos \pi + i e^{\pi} \sin \pi - 1) \).

Since \( \frac{1}{1+i} = \frac{(1-i)}{2} \), we have that this is \( \frac{1}{2} (1-i) (-e^{\pi} - 1) \).

Thus, \( \int_{0}^{\pi} e^{x} \cos x dx = -\frac{(1+e^{\pi})}{2} \) and \( \int_{0}^{\pi} e^{x} \sin x dx = \frac{(1+e^{\pi})}{2} \).

7. Apply the inequality \( \left| \int_{a}^{b} w(t) dt \right| \leq \int_{a}^{b} |w(t)| dt \) (a \( \leq \) b) to show that for all values of \( x \) in the interval \(-1 \leq x \leq 1\), the functions \( P_{n} (x) = \frac{1}{\pi} \int_{0}^{\pi} (x + i\sqrt{1 - x^2} \cos \theta)^n \, d\theta \) (\( n = 0, 1, 2, \ldots \)) satisfy the inequality \( |P_{n} (x)| \leq 1 \).

Solution:
\[
P_{n} (x) = \frac{1}{\pi} \int_{0}^{\pi} (x + i\sqrt{1 - x^2} \cos \theta)^n \, d\theta
\]
\[
|P_{n} (x)| = \left| \frac{1}{\pi} \int_{0}^{\pi} (x + i\sqrt{1 - x^2} \cos \theta)^n \, d\theta \right|
\]
\[
|P_{n} (x)| \leq \frac{1}{\pi} \int_{0}^{\pi} \left| x + i\sqrt{1 - x^2} \cos \theta \right|^n \, d\theta
\]
Consider \( x + i\sqrt{1 - x^2} \cos \theta = \gamma e^{i\theta} \)
\[
\gamma = \sqrt{x^2 + (\sqrt{1 - x^2} \cos \theta)^2} = \sqrt{x^2 + (1 - x^2) \cos^2 \theta}
\]
\[ \theta = \tan^{-1} \left[ \frac{\sqrt{1-x^2} \cos \theta}{x} \right] = \tan^{-1} \theta \]

Let \( \theta' = \frac{\sqrt{1-x^2} \cos \theta}{x} \).

\[ |x + i\sqrt{1-x^2} \cos \theta|^n = |(x + i\sqrt{1-x^2} \cos \theta)^n| = |(x + i\sqrt{1-x^2} \cos \theta)|^n \]
\[ \therefore |e^{i\theta}| = 1 \]

The equation \( |P_n(x)| \leq \frac{1}{\pi} \int_0^\pi \left| \sqrt{x^2 + (1-x^2) \cos^2 \theta} \right|^n \, d\theta \)
\[ \Rightarrow |P_n(x)| \leq \left| \sqrt{x^2 + (1-x^2) \cos^2 \theta} \right|^n \]
\[ = \frac{1}{\pi} \int_0^\pi \left| \sqrt{x^2 \sin^2 \theta + \cos^2 \theta} \right|^n \, d\theta \]
\[ = \frac{1}{\pi} \int_0^\pi \, d\theta \]
\[ = 1. \]

\[ |P_n(x)| \leq 1. \]

Exercises Pages 120-122

1. Show that if \( w(t) = iv(t) \) is continuous on an interval \( a \leq t \leq b \), then
   
   (a) \[ \int_a^b w(-t) \, dt = \int_a^b w(\tau) \, d\tau; \]
   
   (b) \[ \int_a^b w(t) \, dt = \int_a^b \phi(\tau) \, d\tau, \]
   where \( \phi(\tau) \) is the function in equation (9), Section 38.
   
   Suggestion: These identities can be obtained by noting that they are valid for real-valued functions of \( t \).
   
   Solution: Start by writing \( I = \int_a^b w(-t) \, dt = \int_a^b u(-t) \, dt + i \int_a^b v(-t) \, dt. \)
   
   The substitution \( s = -t \) in each of these two integrals on the right then yields
   
   \[ I = -a^b u(s) \, ds - b^a v(s) \, ds = \int_a^b w(s) \, ds. \]
   
   That is, \( I = \int_a^b w(-t) \, dt = \int_a^b w(s) \, ds. \)

   b) Start with \( I = \int_a^b w(-t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt, \)
   and this time the substitution \( t = \phi(s) \) in each of the integrals on the right gives the result.

2. Let \( C \) denote the right-hand half of the circle \( |z| = 2 \), in the counterclockwise direction, and note that two parametric representations for \( C \) are \( z = z(\theta) = 2e^{i\theta} \) \( \left( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right) \) and \( z = Z(y) = \sqrt{4 - y^2} + iy \) \( (-2 \leq y \leq 2). \)
   
   Verify that \( Z(y) = z[\phi(y)] \), where \( \phi(y) = \arctan \frac{y}{\sqrt{4-y^2}} \)

   \(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2} \). Also show that this function \( \phi \) has a positive derivative, as required in the condition following equation (9), Section 38.

   Solution:
   
   We check

   \[ \theta = \tan^{-1} \left[ \frac{\sqrt{1-x^2} \cos \theta}{x} \right] = \tan^{-1} \theta \]
\[ z [\phi (y)] = 2 e^{i \arctan \frac{y}{\sqrt{4-y^2}}} = 2 \left( \cos \arctan \frac{y}{\sqrt{4-y^2}} + i \sin \arctan \frac{y}{\sqrt{4-y^2}} \right). \]

Notice that \( \arctan \frac{y}{\sqrt{4-y^2}} \) is the angle made from the right triangle with base \( \sqrt{4-y^2} \) and height \( y \). Thus, the hypotenuse is \( 2 \). Using this triangle, we may evaluate the \( \cos \) and \( \sin \) to see that \( z [\phi (y)] = 2 \frac{\sqrt{4-y^2}}{2} + i \frac{y}{2} = Z (y) \) as desired. Noting that \( \phi' (y) = \frac{1}{1+ \left( \frac{y}{\sqrt{4-y^2}} \right)^2} \frac{\sqrt{4-y^2} + \frac{y^2}{4-y^2}}{4-y^2} = \frac{1}{\sqrt{4-y^2}} \) we see that the derivative is indeed positive for \( y \in (-2, 2) \).

5. Suppose that a function \( f(z) \) is analytic at a point \( z_0 = z (t_0) \) lying on a smooth arc \( z = z (t) \) \((a \leq t \leq b)\). Show that if \( w (t) = f [z (t)] \), then \( w' (t) = f'' [z (t)] z' (t) \) when \( t = t_0 \). Suggestion: Write \( f(z) = w(x,y) + iv(x,y) \) and \( z(t) = x(t) + iy(t) \), so that \( w(t) = u [x(t), y(t)] + iv [x(t), y(t)] \). Then apply the chain rule in calculus for functions of two real variables to write \( w' = (u_x x' + u_y y') + i (v_x x' + v_y y') \), and use the Cauchy-Riemann equations.

Solution: \( f(z) = u (x(y), y) + iv (x(y), y) \) and \( z(t) = x(t) + iy(t) \) so that \( w(t) = u [x(t), y(t)] + iv [x(t), y(t)] \). The chain rule says that \( w' (t) = u_x x'(t) + u_y y'(t) + i [v_x x'(t) + v_y y'(t)] \). Since \( u_x = v_y \) and \( u_y = -v_x \) we get \( w' (t) = u_x x'(t) - v_x y'(t) + i [v_x x'(t) + u_x y'(t)] = u_x z'(t) + iv_x z'(t) = f' [z (t)] z' (t) \).

6. (a) Let \( y(x) \) be a real-valued function defined on the interval \( 0 \leq x \leq 1 \) by means of the equations \( y (x) = \begin{cases} x^3 \sin \left( \frac{x}{2} \right) & \text{when } 0 < x \leq 1, \\ 0 & \text{when } x = 0. \end{cases} \)

Show that the equation \( z = x + iy \) \((0 \leq x \leq 1)\) represents an arc \( C \) that intersects the real axis at the points \( z = 1/n \) \((n = 1, 2, \ldots)\) and \( z = 0 \).

(b) Let \( y(x) \) be a real-valued function defined on the interval \( 0 \leq x \leq 1 \) by means of the equations \( y (x) = \begin{cases} x^3 \sin \left( \frac{x}{2} \right) & \text{when } 0 < x \leq 1, \\ 0 & \text{when } x = 0. \end{cases} \)

Verify that the arc \( C \) in part (a) is, in fact, a smooth arc. Suggestion: To establish the continuity of \( y(x) \) at \( x = 0 \), observe that \( 0 \leq |x^3 \sin \left( \frac{x}{2} \right)| \leq x^3 \) when \( x > 0 \). A similar remark applies in finding \( y'(0) \) and showing that \( y'(x) \) is continuous at \( x = 0 \).

Solution:

(a) For a complex variable we take the x-axis as the real axis and the y-axis as the imaginary axis. If an arc intersects the real axis, If an arc intersects the real axis, then \( y = 0 \). Given \( z = x + iy (x) \) and \( y (x) = x^3 \sin \left( \frac{\pi}{x} \right) \). When \( y = 0 \), we have \( x^3 \sin \left( \frac{\pi}{x} \right) \).

\[ \Rightarrow x^3 = 0 \text{ or } \sin \left( \frac{\pi}{x} \right) = 0 \]
\[ \Rightarrow x = 0 \text{ or } \frac{\pi}{x} = n \pi, \text{ for } n = 1, 2, \ldots \]
When $x = 0$, the equation $z = x + iy$ becomes $z = 0 + iy(0) \Rightarrow z = 0$.
When $x = \frac{1}{n}$, the equation $z = x + iy$ becomes $z = \frac{1}{n} + i(0) \Rightarrow z = \frac{1}{n}$.

(b) $y(x) = \begin{cases} x^2 \sin \left(\frac{x}{2}\right), & 0 < x \leq 1 \\ 0, & \text{when } x = 0 \end{cases}$

$y'(x) = 3x^2 \sin \left(\frac{x}{2}\right) - x \cos \left(\frac{x}{2}\right) = x \left[3x \sin \left(\frac{x}{2}\right) - \cos \left(\frac{x}{2}\right)\right]$

$|y'(x)| \geq 0$

$|x \left[3x \sin \left(\frac{x}{2}\right) - \cos \left(\frac{x}{2}\right)\right]| \geq 0$

$|x| |3x \sin \left(\frac{x}{2}\right) - \cos \left(\frac{x}{2}\right)| \geq 0$

$|x| \geq 0$ or $|3x \sin \left(\frac{x}{2}\right) - \cos \left(\frac{x}{2}\right)| \geq 0 \Rightarrow$

$0 \leq |3x \sin \left(\frac{x}{2}\right) - \cos \left(\frac{x}{2}\right)| \leq 3x - 1$ when $x > 0$.

$|\cdot| \left|\sin \left(\frac{x}{2}\right)\right| \leq 1$ and $|\cos \left(\frac{x}{2}\right)| \leq 1$.

Exercises Pages 128-130

For the function $f$ and contours $C$ in Exercises 1 through 6, use parametric representations for $C$, or legs of $C$, to evaluate $\int_C f(z) \, dz$.

1. $f(z) = \frac{e^{2i\theta}}{2}$ and $C$ is
   (a) the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$);
   (b) the semicircle $z = 2e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$);
   (c) the circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq 2\pi$).

Solution (1a): $z = 2e^{i\theta}$ so $z'(\theta) = 2ie^{i\theta} \, d\theta$. Thus the integral is

$$\int_0^{2\pi} 2e^{i\theta} + 2ie^{i\theta} \, d\theta = 2e^{i(\pi - 0)} + 2\pi i = -4 + 2\pi i.$$

Solution (1b): $\int_C f(z) \, dz = \int_0^{2\pi} \frac{2e^{i\theta} + 2ie^{i\theta}}{2e^{i\theta} - 2e^{0} + 2\pi i} \, d\theta = \frac{\pi}{2} - 2e^{i\pi} - 2e^{0} - 2\pi i = -4 + 2\pi i.$

Solution (1c): This integration is the sum of the integrations in (a) and (b). So it is equal to $(-4 + 2\pi i) + (4 + 2\pi i) = 4\pi i$.

2. $f(z) = z - 1$ and $C$ is the arc form $z = 0$ to $z = 2$ consisting of
   (a) the semicircle $z = 1 + e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$);
   (b) the segment $0 \leq x \leq 2$ of the real axis.

Solution (2a): $z' = ie^{i\theta}$ so the integral is $\int_C f(z) \, dz = \int_0^{2\pi} e^{i\theta} \, dz$. The antiderivative is $e^{2i\theta}/2$ and when we plug in the terms cancel so we get 0.

4. $f(z)$ is defined by the equations $f(z) = \begin{cases} 1 \text{ when } y < 0, \\ 4y \text{ when } y > 0 \end{cases}$ and $C$ is the arc from $z = -1-i$ to $z = 1+i$ along the curve $y = x^3$.

Solution: Here $z(t) = t + it^3$ so $z'(t) = 1 + 3t^2$. Here $-1 \leq t \leq 1$. Our integral is

$$\int_{-1}^{1} (1 + 3t^2) \, dt + \int_{0}^{t} 4t^3 (1 + 3t^2) \, dt = (1 + i) + (1 + 2i) = 2 + 3i.$$
Solution (4): $C$ is parameterized by $z(t) = t + it^3$, $-1 \leq t \leq 1$. If $t < 0$, $\text{Im } z(t) = t^3 < 0$, so $f(z(t)) = 1$. If $t > 0$, $\text{Im } z(t) = t^3 > 0$, so $f(z(t)) = 4 \text{Im } z(t) = 4t^3$. We compute $z'(t) = 1 + 3it^2$. Thus $\int_C f(z) \, dz = \int_{-1}^{1} f(z(t)) \, z'(t) \, dt = \int_{-1}^{0} f(z(t)) \, z'(t) \, dt + \int_{0}^{1} f(z(t)) \, z'(t) \, dt = \int_{-1}^{0} (1 + i) \, dt + \int_{0}^{1} (1 + i) \, dt = (1 + i) + (1 + 2i) = 2 + 3i.$

Problem statement:
Evaluate $\int_C f(z) \, dz$ for $f(z) = \left\{ \begin{array}{ll} 1 & \text{when } y < 0, \\ 0 & \text{when } y > 0, \end{array} \right.$ and $C$ is the arc from $z = -1 + i$ to $z = 1 + i$ along the curve $y = x^3$.

Solution: The curve can be represented parametrically by $(x(t), y(t)) = (t, t^3)$, $-1 \leq t \leq 1$. We call $C_1$ the portion of $C$ where $-1 \leq t \leq 0$ and $C_2$ the portion for $0 \leq t \leq 1$. Then $\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz = \int_{-1}^{0} (1 + i) \, dt + \int_{0}^{1} (1 + i) \, dt = (1 + i) + (1 + 2i) = 2 + 3i.$

6. $f(z)$ is the branch $z^{-1+i} = \exp \left[ (-1 + i) \log z \right]$ (for $|z| > 0, 0 < \arg z < 2\pi$) of the indicated power function, and $C$ is the positively oriented unit circle $|z| = 1$.

Solution: Let $z = re^{i\theta}$. Then $z^{-1+i} = e^{(-1+i)(\log r+i\theta)} = e^{-\log r - i(\theta + \log r - \theta)} = \frac{e^{-i\theta}}{r} [\cos (\log r - \theta) + i \sin (\log r - \theta)].$

The correct branch is obtained by assuming $0 < \theta < 2\pi$.

The circle $C$ can be parameterized as $e^{i\theta}$, $0 \leq \theta < 2\pi$. Therefore $\int_C f(z) \, dz = \int_{0}^{2\pi} e^{-\theta} (\cos \theta - i \sin \theta) (ie^{i\theta}) \, d\theta$ (since $r = 1$ on contour)

$= i \int_{0}^{2\pi} e^{-\theta} \, d\theta$ (by Euler’s formula)

$= i (1 - e^{-2\pi})$

Problem statement: Evaluate $\int_C f(z) \, dz$ for $f(z)$ for the branch $z^{-1+i} = \exp \left[ (-1 + i) \log z \right]$ (for $|z| > 0, 0 < \arg z < 2\pi$) of the indicated power function, and $C$ is the positively oriented unit circle $|z| = 1$.

Solution:
First, $z(t) = e^{it}$, $0 \leq t \leq 2\pi$. Then $\int_C f(z) \, dz = \int_{0}^{2\pi} \exp \left[ (-1 + i) \log e^{it} \right] ie^{it} \, dt$
\[
\begin{align*}
&= \int_{0}^{2\pi} \exp \left[ (1 + i) \log e^{it} \right] i e^{it} dt \\
&= i \int_{0}^{2\pi} e^{-it} dt = -i e^{-i \frac{2\pi}{3}} = i (1 - e^{-2\pi}).
\end{align*}
\]

10. Let \( C_0 \) denote the circle \(|z - z_0| = R\), taken counterclockwise. Use the parametric representation \( z = z_0 + Re^{i\theta} \) \((-\pi \leq \theta \leq \pi\) for \( C_0 \) to derive the following integration formulas:

(a) \( \int_{C_0} \frac{dz}{z - z_0} = 2\pi i \);

(b) \( \int_{C_0} (z - z_0)^{n-1} dz = 0 \) \( (n = \pm 1, \pm 2, \cdots) \).

Solution: Let \( z = z_0 + Re^{i\theta} \) so that \( dz = Re^{i\theta} d\theta \) and \( z - z_0 = Re^{i\theta} \). Then the first integral is \( \int_{0}^{2\pi} i d\theta = 2\pi i \) and the second integral is \( \int_{0}^{2\pi} R^n e^{ni\theta} d\theta = 0 \).

11. Use the parametric representation in Exercise 10 for the oriented circle \( C_0 \) there to show that \( \int_{C_0} (z - z_0)^{n-1} dz = i \frac{2\pi R^n}{a} \sin (a\pi) \), where \( a \) is any real number other than zero and where the principal branch of the integrand and the principal value of \( R^a \) are taken. [Note how this generalizes Exercise 10(b).]

Solution: \( \int_{C_0} (z - z_0)^{n-1} dz \\
= \int_{-\pi}^{\pi} \exp \left[ (1 - 1) \ln (z - z_0) \right] dz \\
= \int_{-\pi}^{\pi} \exp \left[ (a - 1) \ln (Re^{i\theta}) \right] i Re^{i\theta} d\theta \\
= \int_{-\pi}^{\pi} \exp \left[ (a - 1) \ln R + i\theta \right] i Re^{i\theta} d\theta \\
= i R^a \int_{-\pi}^{\pi} e^{ia\theta} d\theta = \frac{R^a}{a} e^{ia\theta}|_{-\pi}^{\pi} = \frac{R^a}{a} [e^{ia\pi} - e^{ia(-\pi)}] = \frac{R^a}{a} [2i \sin (a\pi)].
\]

At the last step, we are using \( \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \).

Exercises Pages 133-134

1. Without evaluating the integral, show that \( \left| \int_{C} \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3} \) when \( C \) is the same arc as the one in Exercise 1, Section 41.

Solution: On the circle \(|z^2 - 1| \geq 3 \) so the integral is bounded by \( \frac{1}{\pi} \int_{C} |dz| \).

The integral is arclength which is \( 4\pi/4 = \pi \).
Solution (1): On the contour $C$, $|z| = 2$, so $|z^2 - 1| \geq |z^2| - 1 = 2^2 - 1 = 3$, which implies $\left| \frac{1}{z^2 - 1} \right| \leq 1/3$. We already know that $L(C) = \pi$, so $\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{1}{3} L(C) = \frac{\pi}{3}$.

2. Let $C$ denote the line segment from $z = i$ to $z = 1$. By observing that, of all the points on that line segment, the midpoint is the closest to the origin, show that the value of the integral tends to zero as $R$ tends to infinity.

Solution: The closest point, the midpoint, is $(1 + i)/2$ at distance $\frac{1}{\sqrt{2}}$ away from 0. Thus $|z|^4 \geq 1/4$ and so the integral is bounded by $4 \int_C |dz| = 4\sqrt{2}$.

3. Show that if $C$ is the boundary of the triangle with vertices at the points 0, 3i, and -4, oriented in the counterclockwise direction (see Figure 47), then

$$\int_C (e^z - \pi) \leq 5 (12) = 60.$$

Solution:
The triangle is a 3-4-5 right triangle, and thus, its length is 12. Note that $|e^z - z| \leq |e^z| + |z| \leq |e^z| + 4$ on $C$ since it is easy to see that -4 is the point on the triangle farthest from the origin. Note also that Re $z \leq 0$ for all points on the triangle. Thus, $|e^z| = |e^z| |e^{iy}| \leq 1$ on the triangle. Thus, the integrand is $\leq 5$ on the triangle. Hence, $\int_C (e^z - \pi) \leq 5 (12) = 60$.

4. Let $C_R$ denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that $\left| \int_{C_R} \frac{2z^2 - 1}{z^2 + 5z^2 + 4} \right| \leq \frac{\pi R (2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$.

Then, by dividing the numerator and denominator on the right here by $R^4$, show that the value of the integral tends to zero as $R$ tends to infinity.

Solution (4): If $z \in C_R$, then $|z| = R$, so $|2z^2 - 1| \leq |2z^2| + |-1| = 2|z|^2 + 1 = 2R^2 + 1$, and $|z^4 + 5z^2 + 4| = |z^2 + 1| |z^2 + 4| \geq \left( |z|^2 - 1 \right) \left( |z|^2 - 4 \right) = (R^2 - 1)(R^2 - 4) > 0$. Thus $\left| \frac{2z^2 - 1}{z^2 + 5z^2 + 4} \right| = \frac{|2z^2 - 1|}{|z^2 + 5z^2 + 4|} \leq \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}$. We may parametrize $C_R$ by $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$. Then $L(C_R) = \int_0^\pi |z'(\theta)| \, d\theta = \int_0^\pi \pi R e^{i\theta} \, d\theta = \int_0^\pi R \, d\theta = R\pi$. Thus

$$\int_{C_R} \frac{2z^2 - 1}{z^2 + 5z^2 + 4} \leq \frac{\pi R (2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} L(C_R) = \frac{\pi R (2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Solution: The contour $C$ in the upper half of the circle $|z| = R$.

On the contour, $|2z^2 - 1| \leq |2z^2| + 1 = 2R^2 + 1$. Similarly, $|z^4 + 5z^2 + 4| \geq |z^4| = |5z^2| - 4 = R^4 - 5R^2 - 4$. We have
\[
\left| \frac{(2z^2+1)}{(z^2+3z+4)} \right| \leq \frac{(2R^2+1)}{(R^2-3R^2-4)}
\]

for \( z \) on the contour \( C \).

Therefore the contour integral is bounded by \( \pi R \left[ \frac{(2R^2+1)}{(R^2-3R^2-4)} \right] \).

The limit of this bound is 0 as \( R \to \infty \).

5. Let \( C_R \) be the circle \( |z| = R \) \((R > 1)\), described in the counterclockwise direction. Show that \( \left| \int_{C} \log z \, dz \right| < 2\pi \left( \frac{\pi + \ln R}{R} \right) \), and then use l’Hospital’s rule to show that the value of this integral tends to zero as \( R \) tends to infinity.

Solution:

Let us compute the integral. We use the expression \( z = Re^{i\theta} \) for complex numbers belonging to \( C_R \). We note that

\[
\int_{|z|=R} \frac{\log z}{z^2} \, dz = 2\pi \int_{0}^{2\pi} e^{i\theta} \, d\theta
\]

\[
= 2\pi \int_{0}^{2\pi} (\ln R + i(\theta + 2k\pi)) e^{-i\theta} \, d\theta
\]

\[
= -\frac{2\pi}{i} \int_{0}^{2\pi} e^{-i\theta} \, d\theta
\]

\[
= -2\pi i
\]

The limit as \( R \to \infty \) of \( \frac{4\pi i}{R} \) is 0.

6. Let \( C_{\rho} \) denote the circle \( |z| = \rho \) \((0 < \rho < 1)\), oriented in the counterclockwise direction, and suppose that \( f(z) \) is analytic in the disk \( |z| \leq 1 \). Show that if \( z^{-1/2} \) represents any particular branch of that power of \( z \), then there is a non-negative constant \( M \), independent of \( \rho \), such that \( \left| \int_{C_{\rho}} \frac{dz}{z} \right| \leq 2\pi M \sqrt{\rho} \).

Thus show that the value of the integral here approaches 0 as \( \rho \) tends to 0.

Suggestion: Note that since \( f(z) \) is analytic, and therefore continuous, throughout the disk \( |z| \leq 1 \), it is bounded there (Section 17).

Solution:

Since \( f(z) \) is analytic, and therefore continuous throughout the disk \( |z| \leq 1 \), it is bounded there. This implies the existence of a constant \( M \) such that \( |f(z)| \leq \frac{M}{\sqrt{\rho}} \) for all \( |z| \leq 1 \). We have

\[
\left| \int_{C_{\rho}} f(z) \, dz \right| \leq \frac{\pi M}{2} \left| \int_{C_{\rho}} z^{-1/2} \, dz \right|
\]

The latter integral is calculated as follows.

\[
\int_{C_{\rho}} z^{-1/2} \, dz = \int_{0}^{2\pi} \rho^{-1/2} e^{-i(\theta+2k\pi)/2} \rho e^{i\theta} \, d\theta
\]

\[
= i\rho^{1/2} e^{-i\pi/2} \int_{0}^{2\pi} e^{i\theta/2} \, d\theta
\]

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\[ = 2p^{1/2}e^{-ik\pi}(e^{i\pi} - 1) \]
\[ = -4\sqrt{\rho}e^{-ik\pi}. \]

Here \( k = 0 \) or \( k = 1 \) depending on the branch of the function \( z^{-1/2} \). In both cases we have
\[ \left| \int z^{-1/2}f(z) \, dz \right| \leq \frac{M}{2} \left| \int z^{-1/2} \, dz \right| \leq 2\pi \sqrt{\rho}M. \]

Hence, as \( \rho \to 0 \) the value of the integral tends to zero as \( w \)

Exercises Pages 141-142
2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:
   (a) \( \int_{i}^{i/2} e^{\pi z} \, dz \);
   (b) \( \int_{0}^{\pi + 2i} \cos \left( \frac{z}{2} \right) \, dz \);
   (c) \( \int_{1}^{3} (z - 2)^{3} \, dz \).

   Solution (2c): An antiderivative is \( (z - 1)^{4}/4 \) so that the integral is evaluated by evaluating at the endpoints which gives \( 1/4 - 1/4 = 0 \).

   Solution (2a): Since \( \frac{1}{\pi}e^{\pi z} \) is an antiderivative of \( e^{\pi z} \), so
   \[ \int_{i}^{i/2} e^{\pi z} \, dz = \frac{1}{\pi} \left( e^{i\pi/2} - e^{i\pi} \right) = (1 + i) / \pi. \]

   Solution (2b): Since \( 2\sin(z/2) \) is an antiderivative of \( \cos(z/2) \), so
   \[ \int_{0}^{\pi + 2i} \cos \left( \frac{z}{2} \right) \, dz = 2\sin \left( \frac{z}{2} \right) \bigg|_{0}^{\pi + 2i} = 2\sin \left( \pi/2 + i \right) = -i \left( e^{i(\pi/2+i)} - e^{-i(\pi/2+i)} \right) = -i \left( ie^{-1} - (-ie) \right) = 1/e + e. \]

   Solution (2c): Since \( \frac{1}{4} (z - 2)^{4} \) is an antiderivative of \( (z - 2)^{3} \), so
   \[ \int_{1}^{3} (z - 2)^{3} \, dz = \frac{1}{4} (z - 2)^{4} \bigg|_{1}^{3} = \frac{1}{4} \left( (3 - 2) - (1 - 2)^{4} \right) = 0. \]

3. Use the theorem in Section 42 to show that \( \int_{C_{0}}^{z-z_{0}} n^{-1} \, dz = 0 \)
   \( (n = \pm 1, \pm 2, \cdots) \) when \( C_{0} \) is any closed contour which does not pass through the point \( z_{0} \). [Compare Exercise 10(b), Section 40.]

   Solution: The fundamental theorem of calculus implies that the integrals are 0. For \( n = 0, 1, 2, \ldots, \) the integrals are 0 even if the contour passes through \( z_{0} \). For \( n = -1, -2, \ldots, \) the integrand is undefined at \( z = z_{0} \). Therefore the contour must not pass through \( z_{0} \).

4. Find an antiderivative \( F_{2}(z) \) of the branch \( f_{2}(z) \) of \( z^{1/2} \) in Example 4, Section 43, to show that the integral (6) there has value \( 2\sqrt{3}(-1 + i) \). Note that the value of the integral of the function (5) around the closed contour \( C_{2} - C_{1} \) in that example is, therefore, \(-4\sqrt{3}\).
Solution: We want to find \( \int_{C_2} \sqrt{z} \, dz \) with \( \sqrt{z} = \sqrt{r} e^{i \theta/2} \) for \( z = re^{i \theta} \) with \( 0 \leq \theta < 2\pi \).

The branch cut of this branch of \( \sqrt{z} \) can be moved to the positive imaginary axis in such a way that \( \sqrt{z} \) does not change on the contour \( C_2 \). Define \( \sqrt{z} = \sqrt{r} e^{i \theta/2} \) for \( z = re^{i \theta} \) with \( \pi/2 \leq \theta < 5\pi/2 \). The antiderivative of this branch of \( \sqrt{z} \) is given by \[ \frac{2}{3} z^{3/2} = \frac{2}{3} r^{3/2} \cos(\frac{3}{2} \theta) \] for \( z = re^{i \theta} \) with \( 0 \leq \theta < 2\pi \). The integral over the contour \( C_2 \) can be obtained as follows:

\[
\left. \frac{2}{3} z^{3/2} \right|_{-1}^1 = \frac{2}{3} \cdot 3^{3/2} e^{i3\pi/2} \theta = 2 \pi = \frac{2}{3} 3^{3/2} \left( e^{i3\pi} - e^{i3\pi/2} \right) = 2\sqrt{3} (-1 + i) .
\]

5. Show that \( \int_{-1}^1 z^i \, dz = \frac{1 + e^{-\pi}}{2} (1 - i) \), where \( z^i \) denotes the principal branch \( z = \exp(\pi i z) \) \((|z| > 0, -\pi < \text{Arg } z < \pi)\) and where the path of integration is any contour from \( z = -1 \) to \( z = 1 \) that, except, for its end points, lies above the real axis.

Suggestion: Use an antiderivative of the branch \( z^i = \exp(\pi i z) \) \((|z| > 0, -\pi < \text{Arg } z < \pi)\) of the same power function.

Solution: An antiderivative is \( \frac{z^{i+1}}{i+1} \). Evaluating at the endpoints we get \( \frac{1^{i+1}}{(1+i)} - \frac{(-1)^{i+1}}{(1+i)} \). The first term is \( 1/(i + 1) \). The second is \( e^{(i + 1) \log(-1)}/(1 + i) = e^{(i + 1)i}/(1 + i) = -e^{-\pi}/(1 + i) \). Putting it together we get \[ \frac{1 + e^{-\pi}}{2} (1 - i) \].

Solution: The antiderivative is given by \( \frac{1}{1+i} \, z^{(1+i)} = \frac{1}{1+i} e^{(1+i)\text{Log } z} \), with \( \text{Log } (z) \) being the principal branch of the logarithm. Of course, \( \text{Log } (1) = 0 \). Since the contour varies on the upper half of the complex plane before terminating at -1, we can take \( \text{Log } (-1) = i\pi \). By the fundamental theorem of calculus, the integral is equal to \[ \frac{1}{1+i} \left[ e^0 - e^{i\pi(1+i)} \right] = \frac{1 + e^{-\pi}}{2} (1 - i) \].

Exercises Pages 153-156

1. Apply the Cauchy-Goursat theorem to show that \( \int_{C} f(z) \, dz = 0 \) when the contour \( C \) is the circle \( |z| = 1 \), in either direction, and when

(a) \( f(z) = \frac{z^2}{z-3} \);
(b) \( f(z) = \frac{z}{z-1} \);
(c) \( f(z) = \frac{z}{z^2 + 2} \);
(d) \( f(z) = \sec h \, z \);
(e) \( f(z) = \tan z \);
(f) \( f(z) = \text{Log } (z + 2) \).

Solution:
According to the Cauchy-Goursat theorem if \( f(z) \) is an analytic single-valued function in the convex domain \( G \) than for any regular closed curve \( C \) contained in \( G \) the following integral vanishes: 
\[
\int_C f(z) \, dz = 0.
\]

(a) In our case \( f(z) = z^2 / (z - 3) \). This function has the only singularity at point \( z = 3 \). Throughout the disk \( |z| \leq 2 \) the function \( f(z) \) is analytic and single-valued. Hence, according to the Cauchy-Goursat theorem, its integral over the curve \( C : |z| = 1 \) is equal to zero: 
\[
\int_{|z|=1} \frac{z^2 \, dz}{(z-3)} = 0.
\]

(b) As before, the function \( f(z) = ze^{-z} \) is analytic and single-valued throughout the disk \( |z| \leq 2 \): Hence, by the Cauchy-Goursat theorem 
\[
\int_C ze^{-z} \, dz = 0.
\]

(c) The function \( f(z) = 1/ (z^2 + 2z + 2) = (z + 1 + i)(z + 1 - i) \). Hence, the function \( f(z) \) has singularities only at points \( z = -1 - i \) and \( z = -1 + i \). Both points lie outside of the disk \( |z| \leq 1.2 \) since \( | -1 \pm i | = \sqrt{2} \approx 1.4 \). Hence, the function is analytic throughout the disk \( |z| \leq 1.2 \) and by the Cauchy-Goursat theorem 
\[
\int_C \frac{dz}{(z^2 + 2z + 2)} = 0.
\]

(d) The function \( f(z) = \sec hz \) does not have singularities and is analytic in any finite disk \( |z| \leq R \). Hence, by the Cauchy-Goursat theorem 
\[
\int_{|z|=1} \sec hz \, dz = 0.
\]

(e) The function \( f(z) = \tan z = \frac{\sin z}{\cos z} \) is not defined only if the denominator is zero; that is \( \cos z = 0 \). The general solution is \( z = (2n + 1) \frac{\pi}{2} \), where \( n \) is an integer. Whatever the values of \( n \) may be, the value of \( z \) will not lie inside the unit circle. So \( \tan z \) is analytic throughout the region. Since \( f(z) \) is analytic inside and on \( C \), the integral of \( f(z) \) over \( C \) is zero by the Cauchy-Goursat theorem.

(f) The function \( f(z) = \log (z + 2) \) has a singular point \( z = -2 \). It is analytic throughout the disk \( |z| \leq 1.5 \), and by the Cauchy-Goursat theorem 
\[
\int_{|z|=1} \log (z + 2) \, dz = 0.
\]

2. Let \( C_1 \) denote the positively oriented circle \( |z| = 4 \) and \( C_2 \) the positively oriented boundary of the square whose sides lie along the lines \( z = \pm 1, y = \pm 1 \) (Figure 61). With the aid of Corollary 2 in Section 46, point out why 
\[
\int_C f(z) \, dz = \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.
\]

(a) \( f(z) = \frac{1}{3z^2 + 1} \); 
(b) \( f(z) = \frac{z^2}{\sin(z/2)} \); 
(c) \( f(z) = \frac{z}{1 - e^z} \).

Solution (2a): The singularities of this function are at \( \sqrt{-1/3} \) which are inside \( C_2 \). Therefore by Cauchy for doubly connected domains, 
\[
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.
\]
Solution (2a): Note that \( f \) is analytic in \( C \setminus \{ \pm \frac{i}{\sqrt{3}} \} \); and \( \pm \frac{i}{\sqrt{3}} \) are all interior to \( C_2 \). Both \( C_1 \) and \( C_2 \) are positively oriented, so \( \int_{C_1} f dz = \int_{C_2} f dz \).

Solution (2b): Note that \( f \) is analytic in \( C \setminus \{ 2n\pi : n \in \mathbb{N} \} \). When \( n = 0 \), \( 2n\pi = 0 \) lies inside \( C_2 \). When \( n \neq 0 \), \( |2n\pi| \geq 2\pi > 4 \), so \( 2n\pi \) lies outside \( C_1 \). Thus \( \int_{C_1} f dz = \int_{C_2} f dz \).

Solution (2c): Note that \( f \) is analytic in \( C \setminus \{ 2n\pi i : n \in \mathbb{N} \} \). When \( n = 0 \), \( 2n\pi i = 0 \) lies inside \( C_2 \). When \( n \neq 0 \), \( |2n\pi| \geq 2\pi > 4 \), so \( 2n\pi i \) lies outside \( C_1 \). Thus \( \int_{C_1} f dz = \int_{C_2} f dz \).

3. If \( C_0 \) denotes a positively oriented circle \( |z - z_0| = R \), then

\[
\int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 
0 \text{ when } n = \pm 1, \pm 2, \cdots, \\
2\pi i \text{ when } n = 0.
\end{cases}
\]

according the Exercise 10, Section 40. Use that result and Corollary 2 in Section 46 to show that if \( C \) is the boundary of the rectangle

\[0 \leq x \leq 3, 0 \leq y \leq 2, \text{ described in the positive sense, then}\]

\[
\int_{C} (z - 2 - i)^{n-1} dz = \begin{cases} 
0 \text{ when } n = \pm 1, \pm 2, \cdots, \\
2\pi i \text{ when } n = 0.
\end{cases}
\]

Solution (3): Note that \( 2+i \) lies inside \( C \). Let \( C_0 \) be a positively oriented circle \( \{ |z - (2 + i)| = 1/2 \} \). Then \( C_0 \) lies inside \( C \) and \( (z - (2 + i))^{n-1} \) is analytic between \( C_0 \) and \( C \) for any \( n \in \mathbb{N} \). Thus

\[
\int_{C} (z - 2 - i)^{n-1} dz = \int_{C_0} (z - 2 - i)^{n-1} dz = \begin{cases} 
0 \text{ when } n \neq 0, \\
2\pi i \text{ when } n = 0.
\end{cases}
\]

Solution: Using the statement of Exercise 40.20 and Corollary 2 in section 46 we only have to show that we can take a positively circle centered at \( z_0 = 2+i \) such that \( f(z) = (z - z_0)^{n-1} \) is closed on this circle, on the contour \( C \), and on the region between them. If the exponent \( n-1 \) is positive or equal to 0, then \( f \) is entire and we can take this circle to be any circle that is large enough. If the exponent is negative, then \( f \) has a singular point in \( 2+i \). This point is contained in the interior of the rectangle given by \( C \), so again we may take our circle to be a huge circle enclosing this rectangle, then \( f \) has the desired property of being analytic outside the interior of the rectangle and we may apply the corollary.

4. Use the method described below to derive the integration formula

\[
\int_0^\infty e^{-x^2} \cos 2bxdx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).
\]

(a) Show that the sum of the integrals of \( \exp(-x^2) \) along the lower and upper horizontal legs of the rectangular path in Figure 62 can be written

\[
2 \int_0^b e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bxdx
\]

and that the sum of the integrals along the vertical legs on the right and left can be written

\[
-ic^{-a^2} \int_0^b e^{i2ay} dy - ie^{-a^2} \int_0^b e^{-i2ay} dy.
\]
Thus, with the aid of the Cauchy-Goursat theorem, show that
\[ \int_{0}^{a} e^{-x^2} \cos 2bxdx = e^{-b^2} \int_{0}^{a} e^{-x^2} dx + e^{-(a^2+b^2)} \int_{0}^{b} e^{y^2} \sin 2aydy. \]

(b) By accepting the fact that \( \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \) and observing that
\[ \left| \int_{0}^{b} e^{y^2} \sin 2aydy \right| < \int_{0}^{b} e^{y^2} dy, \]
so that the sum of the two integrals can be expressed as
\[ e^{-b^2} \int_{0}^{a} e^{-x^2} dx + e^{-(a^2+b^2)} \int_{0}^{b} e^{y^2} \sin 2aydy. \]

Notice that \( e^{-x^2} \sin (2bx) \) is an odd function of \( x \) while \( e^{-x^2} \cos (2bx) \) are even functions of \( x \). Therefore, the sum of the two integrals can be expressed as
\[ 2 \int_{0}^{a} e^{-x^2} dx - 2e^{b^2} \int_{0}^{a} e^{-x^2} \cos (2bx) dx. \]

The left and right legs of the contour in Figure 62 can be parameterized as
\[ z = a + iy \text{ with } y \text{ increasing from } 0 \text{ to } b \text{ and as } z = -a + iy \text{ with } y \text{ decreasing from } b \text{ to } 0. \]

If Euler’s formula is used for \( e^{-2ay} \) and \( e^{2ay} \), we get
\[ 2e^{-a^2} \int_{0}^{b} e^{y^2} \sin (2ay) dy \]
for the sum of the integrals along the right and left legs.

Since \( e^{-z^2} \) is analytic on the contour shown in Figure 62 and everywhere inside it, Cauchy’s theorem implies that the sum of the integrals
\[ 2 \int_{0}^{a} e^{-x^2} dx - 2e^{b^2} \int_{0}^{a} e^{-x^2} \cos (2bx) dx \text{ and } 2e^{-a^2} \int_{0}^{b} e^{y^2} \sin (2ay) dy \]
is 0. Therefore
\[ \int_{0}^{a} e^{-x^2} \cos (2bx) dx = e^{-b^2} \int_{0}^{a} e^{-x^2} dx + \frac{1}{2} e^{-(a^2+b^2)} + \frac{b}{0} e^{y^2} \sin (2ay) dy. \]

For part (b), we merely have to note that the third term in \( \int_{0}^{a} e^{-x^2} \cos (2bx) dx = \)
\[ e^{-b^2} \int_0^a e^{-x^2} \, dx + e^{-(a^2+b^2)} + \int_0^b e^{y^2} \sin (2ay) \, dy \] above tends to 0 as \( a \to \infty \) for any \( b > 0 \).

6. Let \( C \) denote the positively oriented boundary of the half disk \( 0 \leq r \leq 1, 0 \leq \theta \leq \pi \), and let \( f(z) \) be a continuous function defined on that half disk by writing \( f(0) = 0 \) and using the branch \( f(z) = \sqrt{r} e^{i\theta/2} \) \( (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}) \) of the multiple-valued function \( z^{1/2} \). Show that \( \oint_C f(z) \, dz = 0 \) by evaluating separately the integrals of \( f(z) \) over the semicircle and the two radii which make up \( C \). Why does the Cauchy-Goursat theorem not apply here?

Solution: The branch of \( z^{1/2} \) which is taken to be \( f(z) \) is analytic everywhere except on the negative imaginary axis and at \( z = 0 \). The contour \( C \) passes through \( z = 0 \), a point where \( f(z) \) is not analytic. Therefore Cauchy’s theorem does not apply. \( C \) can be deformed very slightly by bumping it up a bit near the origin. Then \( \oint f(z) \, dz \) changes only slightly as \( f(z) \) is continuous at \( z = 0 \). Cauchy’s theorem applies over the deformed contour and implies that \( \oint f(z) \, dz = 0 \). By making the bump smaller and smaller and taking the limit, we can conclude that \( \oint_C f(z) \, dz = 0 \).

Exercises Pages 162-164

1. Let \( C \) denote the positively oriented boundary of the square whose sides lie along the lines \( x = \pm 2 \) and \( y = \pm 2 \). Evaluate each of these integrals:

(a) \( \oint_C e^{-z/2} \, dz \);
(b) \( \oint_C \frac{\cos z}{z(z^2 + 8)} \, dz \);
(c) \( \oint_C \frac{z \, dz}{2z + 1} \);
(d) \( \oint_C \frac{\cosh z}{z^2} \, dz \);
(e) \( \oint_C \frac{\tan(z/2)}{(z - z_0)^2} \, dz \) \( (-2 < x_0 < 2) \)

Solution (1a): The function \( e^{-z} \) is analytic so by CIF the integral is \( 2\pi i e^{-\pi i/2} = 2\pi i (-i) = 2\pi \).

Solution (1b): The function \( \cos(z) \) / \( (z^2 + 8) \) is analytic inside the contour so the integral is \( 2\pi i \cos(0) / 8 = \pi i / 4 \).

Solution (1a): Since \( \pi i / 2 \) lies inside \( C \), and \( \frac{\cos z}{z^2 + 8} \) is analytic inside and on \( C \) (it is analytic throughout \( C \) except at \( \pm \sqrt{8}i \), which both lie inside \( C \)), so
\[ \oint_C \frac{e^{-z}}{z - (\pi i/2)} = 2\pi i e^{-\pi i/2} = 2\pi. \]

Solution (1b): Since 0 lies inside \( C \), and \( \frac{\cos z}{z^2 + 8} \) is analytic inside and on \( C \) (it is analytic throughout \( C \) except at \( \pm \sqrt{8}i \), which both lie outside \( C \)), so
\[ \oint_C \frac{\cos z}{z^2 + 8} \, dz = 2\pi i \frac{\cos(0)}{0^2 + 8} = \frac{\pi i}{4}. \]
Solution (1c): Since -1/2 is inside C and z/2 is analytic in C, so 
\[ \int_C \frac{z}{z-1/2} \, dz = 2\pi i \left( -1/2 / 2 \right) = -\pi i / 2. \]

Solution (1d): Since 0 lies inside C, and \( \cosh \) is analytic in C, so 
\[ \int_C \frac{\cosh z}{z} \, dz = \frac{2\pi i}{3!} \cosh(0) = \frac{\pi i}{3} \sinh(0) = 0. \]

Solution (1e): Since \( x_0 \) lies inside C, and \( \tan(z/2) \) is analytic in C, so 
\[ \int_C \frac{\tan(z/2)}{(z-x_0)^2} \, dz = 2\pi i \frac{d}{dz} \tan(z/2) \big|_{z=x_0} = \pi i \sec^2(x_0/2). \]

2. Find the value of the integral of \( g(z) \) around the circle \( |z-i|=2 \) in the portion sense when
(a) \( g(z) = \frac{1}{z+i}; \)
(b) \( g(z) = \frac{1}{z^2 + 1}. \)

Solution (2a): We have \( \frac{1}{z+i} = \frac{1}{z+2i} \). The first factor is analytic inside the contour so the integral is \( 2\pi i \frac{1}{16} = \frac{\pi i}{8} \).

Solution (2b): We have \( \frac{1}{z^2 + 1} = \frac{1}{(z+2i)(z-2i)} \). So that by CIF for derivatives the integral is \( 2\pi i f'(2i) = \frac{-4\pi i}{(4i)^2} = \frac{\pi}{16} \).

Solution (2a): Let C denote the positively oriented circle \( \{ |z-i|=2 \} \). Note that 2i lies inside C, and -2i does not. So \( \frac{1}{z+2i} \) and \( \frac{1}{(z+2i)^2} \) are analytic inside and on C. 
\[ \int_C \frac{1}{z^2 + 1} \, dz = \int_C \frac{1/(z+2i)}{(z-2i)^2} \, dz = 2\pi i \frac{d}{dz} \left( \frac{1}{z-2i} \right) \big|_{z=2i} = 2\pi i \frac{-2}{(2i+2i)^2} = 2\pi i \frac{-2}{-64} = \pi / 16. \]

Solution (2b): Let C denote the positively oriented circle \( \{ |z-i|=2 \} \). Note that 2i lies inside C, and -2i does not. So \( \frac{1}{z+2i} \) and \( \frac{1}{(z+2i)^2} \) are analytic inside and on C.
\[ \int_C \frac{1}{z^2 + 1} \, dz = \int_C \frac{1/(z+2i)^2}{(z-2i)} \, dz = 2\pi i \frac{d}{dz} \left( \frac{1}{z-2i} \right) \big|_{z=2i} = 2\pi i \frac{-2}{(2i+2i)^2} = 2\pi i \frac{-2}{-64} = \pi / 16. \]

3. Let C be the circle \( |z|=3 \), described in the positive sense. Show that if 
\[ g(w) = \int_C \frac{2z^2 - z - 2}{z-w} \, dz \quad (|w| \neq 3), \] 
then \( g(2) = 8\pi i \). What is the value of \( g(z) \) when \( |w| > 3? \)

Solution (3): Let \( f(z) = 2z^2 - z - 2 \). By CIF \( g(2) = 2\pi i f(2) = 8\pi i \). For the second part since the function is analytic inside the contour when \( |w| > 3 \) the integral is 0 by Cauchy Theorem.

Solution (3): Let \( f(z) = 2z^2 - z - 2 \). Then \( f \) is an entire function. Since 2 lies inside C, so by Cauchy’s integral formula, \( g(2) = \int_C \frac{f(z)}{z-2} \, dz = 2\pi i f(2) = 2\pi i 4 = 8\pi i \). If w is outside C, let \( h_w(z) = f(z) / (z-w) \). Then h is analytic on and inside C. Thus by Cauchy-Goursat theorem, \( g(w) = \int_C h_w(z) \, dz = 0 \).

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write \( g(w) = \int_C \frac{z^3 + 2z}{z-w} \, dz \). Show that \( g(w) = 6\pi i w \) when w is inside C and that g(w) = 0 when w is outside C.

Solution (4): Since \( z^3 + 2z \) is analytic in C, so if w is inside C, we have
bounded throughout C, i.e., there exists a constant M such that
\[(z^3 + 2z) \in C\] is analytic inside and on C. Thus \(g(w) = \int_C \frac{z^3 + 2z}{z-w} \, dz = 0\).

5. Show that if \(f\) is analytic within and on a simple closed contour C and \(z_0\) is not on C, then \(\int_C \frac{f'(z) \, dz}{z-z_0} = \int_C \frac{f(z) \, dz}{z-z_0}^2\).

Solution: The Cauchy Integral Formula applied to \(f'\) gives \(f'(z_0) = \int_C \frac{f(z) \, dz}{z-z_0}\).

The formula for higher derivatives (p.161, (4)) gives \(f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z) \, dz}{(z-z_0)^2}\).

This implies the claim.

6. Let \(f\) denote a function that is continuous on a simple closed contour C.

Prove that the function \(g(z) = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{s-z}\) is analytic at each point \(z\) interior to C and that \(g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z)^2}\) at such a point.

Solution:
In order to prove the analyticity of the function \(g(z)\) we need to demonstrate that it is differentiable. To this end we need to find the linear part of the increment of the function \(g(z) = g(z+h) - g(z)\) for every two points \(z\) and \(z+h\) interior to C. We have

\[
\frac{1}{s-(z+h)} - \frac{1}{s-z} = \frac{h}{(s-z)^2} + h^2 \left(\frac{1}{(s-z)^2} - \frac{1}{(s-z)(s-(z+h))}\right)
\]

From the definition of function g(z) and from the formula above it follows that

\[
g(z+h) - g(z) = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{s-(z+h)} - \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{s-z} \]

\[
= h \left\{ \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z)^2} \right\} + h^2 \left\{ \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z)^2(s-(z+h))} \right\}.
\]

Now we have to prove that the second term is of order \(O(h^2)\). In other words, we need to limit the second integral in the right-hand side by a constant. Since both points \(z\) and \(z+h\) are interior to C, there exists a constant \(\delta\) such that \(|s-z| > \delta, |s-(z+h)| > \delta\). Besides, since \(f(z)\) is continuous on C it is bounded throughout C, i.e., there exists a constant \(M\) such that \(|f(z)| \leq M\) for each point \(z \in C\). Hence, by the Mean Value Theorem

\[
\left| \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z)^2(s-(z+h))} \right| \leq \frac{ML}{2\pi R^3} < \text{constant}.
\]

Here \(L\) is the length of C. Thus, we obtain that the linear part of the increment of the function \(g(z)\) exists and is given by \(g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z)^2}\). Hence, the proof that the function \(g(z)\) is analytical is complete, and its derivative is given by the required formula.
7. Let C be the unit circle \( z = e^{i\theta} \) \((-z \leq \theta \leq \pi)\). First show that, for any real constant \(a\), \(\int_C \frac{e^{az}}{z} \, dz = 2\pi i\). Then write this integral in terms of \(\theta\) to derive the integration formula \(\int_0^\pi e^{a\cos \theta} \cos (a \sin \theta) \, d\theta = \pi\).

Solution (7): Since \(e^{az}\) is analytic in C and 0 lies inside C, so we have \(\int_C \frac{e^{az}}{z} \, dz = 2\pi i e^{a\theta} = 2\pi i\). From the definition of a line integral, we have
\[
\int_C \frac{e^{az}}{z} \, dz = \frac{\pi}{-\pi} \int \exp(\alpha e^{i\theta}) \, d\theta = i \int \exp(\alpha e^{i\theta}) \, d\theta. \quad \text{Thus} \quad \int \exp(\alpha e^{i\theta}) \, d\theta = 2\pi.
\]
Since \(\exp(\alpha e^{i\theta}) = \exp(a \cos \theta + ia \sin \theta)\),
\[
\Re \left( \int_{-\pi}^{\pi} \exp(\alpha e^{i\theta}) \, d\theta \right) = 2\pi. \quad \text{Since} \quad e^{a \cos(-\theta)} \cos(a \sin(-\theta)) = e^{a \cos \theta} \cos(a \sin \theta), \quad \text{so} \quad \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta =
\]
\[
\int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta. \quad \text{Thus} \quad \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta =
\]
\[
\frac{1}{2} \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta = \pi.
\]

9. Follow the steps below to verify the expression \(f''(z) = \frac{1}{\pi} \int_C \frac{f(s) ds}{(s-z)^2}\) in the lemma in Section 48.

(a) Use the expression for \(f'(z)\) in the lemma to show that
\[
\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi} \int_C \frac{f(s) ds}{(s-z)^2} = \frac{1}{2\pi i} \int_C \frac{2(s-z) - \Delta z}{(s-z)^2} f(s) ds.
\]

(b) Let D and d denote the largest and smallest distances, respectively, from \(z\) to points on C. Also, let \(M\) be the maximum value of \(|f(s)|\) on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of the expression for \(f'(z)\) in the lemma, show that when \(0 < |\Delta z| < d\), the value of the integral on the right-hand side in part (a) is bounded from above by \(\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d-|\Delta z|^2)d^3} L\).

(c) Use the results in parts (a) and (b) to obtain the desired expression for \(f''(z)\).

Solution: In view of the expression for \(f'\) in this lemma.
\[
\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi} \int_C \frac{f(s) ds}{(s-z)^2} = \frac{1}{2\pi i} \int_C \left[ \frac{2(s-z)-\Delta z}{(s-z)^2} - \frac{1}{(s-z)^2} \right] f(s) ds.
\]
Then
\[
\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi} \int_C \frac{f(s) ds}{(s-z)^2} = \frac{1}{2\pi i} \int_C \left[ \frac{2(s-z)-\Delta z}{(s-z)^2} - \frac{2}{(s-z)^2} \right] f(s) ds
\]
\[
= \frac{1}{2\pi i} \int_C \left[ \frac{3(s-z) - 2|\Delta z|^2}{(s-z)^2} \right] f(s) ds.
\]
b) The triangle inequality tells us that
\[ 3(s - z)Δz - 2(Δz)^2 \leq 3|s - z| |Δz| + 2|Δz|^2 \leq 3D |Δz| + 2|Δz|^2. \]
We also can see that \(|s - z - Δz| \geq d - |Δz| > 0\) (p.160, line 3). Hence,
\[ |(s - z - Δz)^2 - 2(Δz)^2| \geq (d - |Δz|)^2 d^3 > 0. \]
Together we obtain
\[ \int_C \left| \frac{3(s-z)Δz-2(Δz)^2}{(s-z-Δz)^2} f(s) \, ds \right| \leq \frac{(3D|Δz|+2|Δz|^2)M}{(d-|Δz|)^2 d^3}. \]

3) If we let \(Δz\) tend to 0 in this inequality, we find
\[ \lim_{Δz \to 0} \frac{1}{2\pi i} \int_C \frac{3(s-z)Δz-2(Δz)^2}{(s-z-Δz)^2} f(s) \, ds = 0. \]
This, together with the result in part a), yields the desired expression for \(f'\).

Exercises Pages 171-173

1. Let \(f\) be an entire function such that \(|f(z)| \leq A|z|\) for all \(z\), where \(A\) is a fixed positive number. Show that \(f(z) = a_1 z\), where \(a_1\) is a complex constant.

**Suggestion:** Use Cauchy's inequality (Section 49) to show that the second derivative \(f''(z)\) is zero everywhere in the plane. Note that the constant \(M_R\) in Cauchy's inequality is less than or equal to \(A(|z_0| + R)\).

Solution: By Cauchy inequality we have \(|f''(z_0)| \leq \frac{2M_R}{R^2}\) where \(M_R\) is the maximum of \(|f(z)|\) on the circle of radius \(R\) centered at \(z_0\). In this case \(|z| \leq |z_0| + R\) so \(M_R \leq A(|z_0| + R)\). Plugging that in we get \(|f''(z_0)| \leq \frac{A(|z_0|+R)}{R^2}\).

As \(R \to \infty\) the right hand side goes to 0 and this says that \(f''(z_0) = 0\). Since this is true for any \(z_0\) we have \(f''(z) = 0\) for all \(z\) and this means \(f'\) is a constant \(a_1\). But this means that \(f(z) = a_1 z + b\). Since \(|f(z)| \leq A|z|\) we have \(b_1 = 0\).

Solution: For \(R > 0\), let \(C_R\) denote the positively oriented circle
\[ \{z \in C : |z| = R\}. \]
Fix \(z_0 \in C\). Then \(z_0\) lies inside \(C_R\) if \(R > |z|\). From Cauchy's formula,
\[ f''(z_0) = \frac{2}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^3} \, dz. \]
For \(z \in C_R\),
\[ \left| \frac{f(z)}{(z-z_0)^3} \right| \leq \frac{AR}{(R-|z_0|)^3}. \]
Thus \(|f''(z_0)| \leq \frac{L(C_R) AR}{(R-|z_0|)^3} \to 0\) as \(R \to \infty\), i.e., \(f''(z_0) = 0\). Since \(z_0\) is chosen from \(C\) arbitrarily, so \(f'(z) = 0\) for all \(z \in C\). Thus \(f'(z)\) is constant in \(C\). Let \(a_1\) be the constant and \(g(z) = f(z) - a_1 z\). Then \(g(z) = 0\) for all \(z \in C\). Thus \(g(z)\) is a constant in \(C\). Let the constant be \(C\). Then \(f(z) = a_1 z + C\) for all \(z \in C\). From \(|f(0)| \leq A0 = 0\) we get \(f(0) = 0\). Thus \(C = 0\) and \(f(z) = a_1 z\).

2. Suppose that \(f(z)\) is entire and that the harmonic function \(u(x,y) = \text{Re}[f(z)]\) has an upper bound \(u_0\); that is, \(u(x,y) \leq u_0\) for all points \((x,y)\) in the xy plane. Show that \(u(x,y)\) must be constant throughout the plane.

**Suggestion:** Apply Liouville's theorem (Section 49) to the function \(g(z) = \exp[f(z)]\).

Solution: Let \(g(z) = e^{f(z)} = e^{u+iv}\). Since \(|e^{f(z)}| = e^u\) and \(u\) is assumed bounded, we have \(e^{f(z)}\) a constant. But this means \(e^u\) constant which means that \(u\) is a constant.
Solution: Let \( g(z) = \exp(f(z)) \). Then \( g \) is entire and \( |g(z)| = \exp(u(z)) \leq u_0 \). From Liouville’s Theorem, \( g \) must be a constant. Thus \( u(z) = \ln(|g(z)|) \) is also a constant throughout \( C \).

4. Let a function \( f \) be continuous in a closed bounded region \( R \), and let it be analytic and not constant throughout the interior of \( R \). Assuming that \( f(z) \neq 0 \) anywhere in \( R \), prove that \( |f(z)| \) has a minimum value \( m \) in \( R \) which occurs on the boundary of \( R \) and never in the interior. Do this by applying the corresponding result for maximum values (Section 50) to the function \( g(z) = 1/f(z) \).

Solution: Let \( g(z) = 1/f(z) \). Since \( f \) is continuous on \( R \), analytic in the interior of \( R \) and \( f(z) \neq 0 \) for all \( z \in R \), so \( g \) is continuous in \( R \) and analytic in the interior of \( R \). Since \( R \) is closed and bounded, \( g \) is continuous, so \( g \) attain its maximum on \( R \). That means there is \( z_0 \in R \) such that \( |g(z)| \leq |g(z_0)| \) for all \( z \in R \). Thus \( |f(z)| \geq |f(z_0)| \) for all \( z \in R \). If \( z_0 \in \text{int} \ R \) then from maximum modulus principle we have \( g \) is a constant in \( \text{int} \ R \). Because \( g \) is continuous, so \( g \) is also constant in \( R \). From \( f(z) = 1/g(z) \) we then have that \( f \) is constant in \( R \) which is a contradiction. Thus \( z_0 \) must lie on the boundary of \( R \).

6. Consider the function \( f(z) = (z + 1)^2 \) and the closed triangular region \( R \) with vertices at the points \( z = 0, z = 2, \) and \( z = i \). Find points in \( R \) where \( |f(z)| \) has its maximum and minimum values, thus illustrating results in Section 50 and Exercise 4.

**Suggestion:** Interpret \( |f(z)| \) as the square of the distance between \( z \) and -1.

Solution: We must have \( |f(z)| \) taking its maximum on the boundary and since it is never 0 also its minimum on the boundary. Since \( |f(z)| \) is the square of the distance between \( z \) and -1 the maximum is taken on at the furthest point from -1 which is at \( z = 2 \). The closest point is at \( z = 0 \).

7. Let \( f(z) = u(x,y) + iv(x,y) \) be a function that is continuous on a closed bounded region \( R \) and analytic and not constant throughout the interior of \( R \). Prove that the component function \( u(x,y) \) has a minimum value in \( R \) which occurs on the boundary of \( R \) and never in the interior. (See Exercise 4.)

Solution???: Suppose \( u \) has a minimum at a point in the interior say at \( u(z_0) \). Let \( f(z) = e^{f(z)} \) which is never 0. We have \( |f(z)| = e^u \). The minimum of this is at \( z_0 \). But this violates (4).

10. Let \( z_0 \) be a zero of the polynomial \( P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \) \( (a_n \neq 0) \) of degree \( n \) \( (n \geq 1) \). Show in the following way that \( P(z) = (z - z_0)Q(z) \), where \( Q(z) \) is a polynomial of degree \( n-1 \).

(a) Verify that \( z^k - z_0^k = (z - z_0) (z^{k-1} + z^{k-2}z_0 + \cdots + z_0^{k-2} + z_0^{k-1}) \) \( (k = 2, 3, \cdots) \).

(b) Use the factorization in part (a) to show that \( P(z) - P(z_0) = (z - z_0)Q(z) \), where \( Q(z) \) is a polynomial of degree \( n-1 \), and deduct the desired results from this.

Solution (10a): We calculate the right-hand side of the equation:
\[(z - z_0) \left( z^{k-1} + z^{k-2}z_0 + \cdots + zz_0^{k-2} + z_0^{k-1} \right) = z^k + z^{k-1}z_0 + \cdots + z^2z_0^{k-2} + zz_0^{k-1} - \left( z^{k-1}z_0 + \cdots + z_0^k \right) = z^k - z_0^k. \]

(10b) Now \( P(z) - P(z_0) = a_0 + \cdots + a_nz^n - (a_0 + \cdots + a_nz_0^n) \)
\[= (a - a_0) + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots + a_n (z - z_0)^n, \]
\[= (z - z_0) \left[ a_1 + a_2 (z - z_0) + \cdots + a_n (z - z_0)^{n-1} \right], \]
\[= (z - z_0) \tilde{Q}(z). \]